Which kind of measures are phase spectral measures?

Xiaoye Fu

Central China Normal University

Joint work with Ting Zhou

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Overview

1. Background
2. Three pure type phase spectral measures
3. Cartesian product phase spectral measures

Spectral theory

Let μ be a Borel probability measure on $\mathbb{R}^d$. We call μ a spectral measure(resp. a (Riesz-)frame spectral measure) if there exists a countable set $\Lambda \subseteq \mathbb{R}^d$ such that

$$E(\Lambda) := \{e^{2\pi i\langle \lambda, x\rangle} : \lambda \in \Lambda\}$$

forms an orthogonal basis(resp. a Riesz basis/a frame) for $L^2(\mu)$. In this case, Λ is called a spectrum(resp. a (Riesz-)frame spectrum) of μ.

Particularly, if $d\mu = \mathbf{1}_\Omega dx$ for a bounded measurable set Ω and μ is a spectral measure, then Ω is called a spectral set, which was presented by Fuglede.

Some progress on spectral measures

Fuglede, 1974

Lattice tiles are spectral sets.

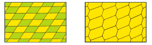


Figure: Convex tiles

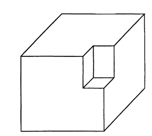


Figure: The notched cube

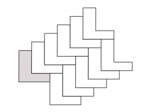


Figure: L-shape region

Triangle and disk are not spectral sets.

Some progress on spectral measures

Jorgensen & Pedersen 1998

Standard $1/4$ Cantor measure μ_4 is a spectral measure, where

$$\mu_4(\cdot) = \frac{1}{2}\mu(4\cdot) + \frac{1}{2}\mu(4\cdot - 2).$$

Standard $1/3$ Cantor measure μ_3 is not a spectral measure, where

$$\mu_3(\cdot) = \frac{1}{2}\mu(3\cdot) + \frac{1}{2}\mu(3\cdot - 2).$$

There are many studies on spectrality and non-spectrality for some measurable sets, self-similar and Moron measures, finite cyclic groups, p-adic fields $\cdots$.

Phase spectral measures

Question 1: Does a triangle or a disk or the $1/3$ cantor measure admit an exponential Riesz basis/a fourier frame? (Not known yet!)

Question 2: For those non-spectral measures μ, Does $L^2(\mu)$ admit an orthogonal basis\ Fourier frame\ Riesz basis of exponential type $E(\Lambda, \varphi) = \{e^{2\pi i \langle \lambda, \varphi(x) \rangle} : \lambda \in \Lambda\}$, where φ is Borel measurable which is called a *phase function* and is not necessarily linear?

Gabardo-Lai-Oussa 2021 Question 2 can be reduced to the push forward measure of μ under φ which is denoted by $\varphi_* \mu$ is a (Riesz-frame) spectral measure.

$$\varphi_* \mu(E) = \mu(\varphi^{-1}(E)), E \subseteq \mathbb{R}^d, \quad E \quad Borel.$$

For convenience, we call μ a phase spectral measure if $\varphi_* \mu$ is a spectral measure.

The characterization of phase spectral measures

Theorem (Gabardo, Lai & Oussa, 2021)

Let μ be a finite Borel measure supported on $K_\mu \subset \mathbb{R}^d$, $\Lambda \subseteq \mathbb{R}^d$ be a countable set and let $\varphi : K_\mu \to \mathbb{R}^d$ be Borel measurable. Then

The collection $E(\Lambda, \varphi)$ is an orthogonal basis/a frame/a Riesz basis for $L^2(\mu)$

$$\Updownarrow$$

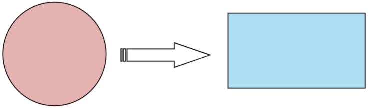
φ is μ- essentially injective and $E(\Lambda)$ is an orthogonal basis/a frame/a Riesz basis for $L^2(\varphi_* \mu)$.

μ-essentially injecive Define μ and φ as above. φ is said to be μ-essentially injective if there exists a Borel set $\mathcal{N} \subseteq K_\mu$ with $\mu(\mathcal{N}) = 0$ such that φ is injective on $K_\mu \setminus \mathcal{N}$.

Phase spectral measures

Gabardo-Lai-Oussa 2021 $\frac{1}{3}$-cantor measure μ_3 is a phase spectral measure.
Define $\varphi : K_3 \to K_4$ by $\varphi(\sum_{i=1}^{\infty} \frac{\epsilon_i}{3^i}) = \sum_{i=1}^{\infty} \frac{\epsilon_i}{4^i}$, $\epsilon_i \in \{0, 2\}$. We have $\varphi * \mu = \nu_4$.

Holhos, 2017 Any p-ball can be transformed to a square under a measure preserving map. So p-ball (of course any disk) is a phase spectral set.



Phase spectral measures

We mainly study the following question raised by Gabardo-Lai-Oussa in 2021.

Question 3: Given any finite Borel measure μ on $\mathbb{R}^d$, is it true that every $L^2(\mu)$ can admit some $E(\Lambda, \varphi)$ as an orthogonal basis? Can we find a Borel measure that does not admit any orthogonal basis with non-linear phase function?

Generalization of Gabardo-Lai-Oussa's result

Define φ as a mapping that can be projected into different dimensions opens up more possibilities for us to construct spectral measures $\varphi_*\mu$.

Theorem (F.-Zhou)

Let μ be a finite Borel measure supported on $K_\mu \subseteq \mathbb{R}^n$, $\Lambda \subseteq \mathbb{R}^d$ be a countable set and let $\varphi : K_\mu \to \mathbb{R}^d$ be a Borel measurable function. Then

$E(\Lambda, \varphi)$ is an orthogonal basis/a frame / a Riesz basis for $L^2(\mu)$ iff φ is $\mu-$essentially injective and $E(\Lambda)$ is an orthogonal basis/a frame/a Riesz basis for $L^2(\varphi_*\mu)$.

- Gabardo-Lai-Oussa's result for $n = d$.

Pure type property of phase spectral measures

Theorem (He-Lai-Lau, 2013)

A spectral measure μ must be of pure type, that is, μ is a discrete measure with finite support, a singularly continuous measure or an absolutely continuous measure with respect to Lebesgue measure.

Proposition (F.-Zhou)

A phase spectral measure is either continuous or discrete. Specifically,

(i) Let μ be a continuous measure supported on $K_\mu \subseteq \mathbb{R}^n$ (i.e. $\mu(x) = 0$ for any single point x) and let $\varphi : K_\mu \to \mathbb{R}^d$ be a μ-essentially injective function. Then $\varphi_*\mu$ is also a continuous measure on $\mathbb{R}^d$.

(ii) Let μ be a (finite) discrete measure supported on $K_\mu \subseteq \mathbb{R}^n$ and let $\varphi : K_\mu \to \mathbb{R}^d$ be a μ-essentially injective function. Then $\varphi_*\mu$ is also a (finite) discrete measure on $\mathbb{R}^d$.

Discrete phase spectral measures

In the following, we will discuss Question 3 in terms of discrete, singularly continuous and absolutely continuous measures. We completely answer Question 3 for discrete measures.

Theorem (F.-Zhou)

Let μ be a discrete measure supported on $K_\mu \subseteq \mathbb{R}^n$. Then
(i) If $\#K_\mu < \infty$, μ is a phase spectral measure iff μ is equally weighted distribution.

(ii) If $\#K_\mu = \infty$, then μ can not be a phase spectral measure.

- $\mu = \sum_{a\in A} p_a \delta_a$ is spectral $\Rightarrow \#A < \infty$ and all $p_a, a \in A$ are the same.

Singularly continuous phase spectral measures

An iterated function system (IFS): a family of contraction $\{F_1, \cdots, F_m\}$ on $\mathbb{R}^d$ associated with probabilities $\{p_1, \cdots, p_m\}$ and $\sum_{i=1}^m p_i = 1$, where $m \geq 2$.

An invariant attractor: $E = \bigcup_{i=1}^m F_i(E)$, which is a non-empty compact set.

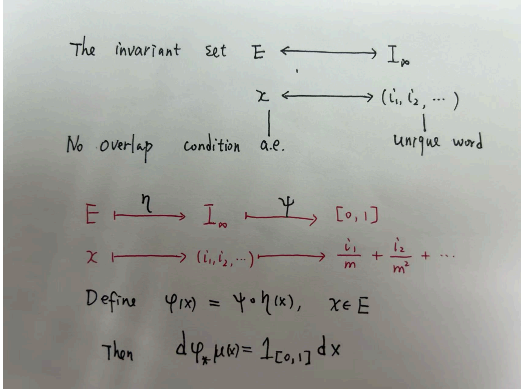
An invariant measure: $\mu(A) = \sum_{i=1}^m p_i \mu(F_i^{-1}(A))$ for all Borel sets $A \subseteq \mathbb{R}^d$.

no-overlap condition: $\mu(F_j(E) \cap F_i(E)) = 0$ for all $1 \leq i < j \leq m$.

Singularly continuous phase spectral measures

Theorem (F.-Zhou)

Let μ be a Borel probability measure generated by an IFS $\{F_0, F_1 \cdots, F_{m-1}\}$ with equal probability weights. If μ satisfies the no-overlap condition, then μ is phase-spectral.



Absolutely continuous phase spectral measures

For an absolutely continuous measure, we have the following partial result.

Theorem (F.-Zhou)

Let μ be an absolutely continuous measure.
(i) If μ is a finite Lebesgue measure restricted on an open set $\Omega \subseteq \mathbb{R}^n$, then μ is a phase spectral measure.

(ii)If μ is a finite and positive absolutely continuous measure supported on $K_\mu \subset \mathbb{R}$, then μ is a phase spectral measure.

Cartesian product spectral measures

For two bounded, measurable sets $A \subseteq \mathbb{R}^n$, $B \subseteq \mathbb{R}^m$, Kolountzakis (2016) posed the following conjecture.

Conjecture (Kolountzakis, 2016)

Let $A \subset \mathbb{R}^n$, $B \subset \mathbb{R}^m$ be two bounded, measurable sets. Then their product $\Omega = A \times B$ is a spectral set if and only if both A and B are spectral sets.

The "if" part of this conjecture is obvious. (Λ_1, A) and (Λ_2, B) are spectral pairs $\Longrightarrow$ $(A \times B, \Lambda_1 \times \Lambda_2)$ is a spectral pair.

The "only if" part is not trivial in the case that Λ does not have a product structure. It was proved in some priori assumption on $A = [0,1]^n$ (Greenfeld-Lev, 2016, 2020).

Cartesian product spectral measures

Generalized Kolountzakis' Conjecture $\mu \times \nu$ is spectral $\Leftrightarrow$ both μ and ν are spectral.

Theorem (F.-Zhou)

Let $d\mu = \mathbf{1}_I dx$ be a Lebesgue measure restricted on a unit cube $I \subseteq \mathbb{R}^n$ and let $d\nu = g(x)dx$ be an absolutely continuous measure on $\mathbb{R}^m$.Then $\mu \times \nu$ is a spectral measure if and only if ν is a spectral measure.

Theorem (F.-Zhou)

Let μ and ν be two finite Borel measures on $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively and let $\Lambda_1 \subseteq \mathbb{R}^n$ and $\Lambda_2 \subseteq \mathbb{R}^m$ be countable sets. Then $\mu \times \nu$ is a spectral measure with a spectrum $\Lambda_1 \times \Lambda_2$ if and only if μ is a spectral measure with a spectrum Λ_1 and ν is a spectral measure with a spectrum Λ_2.

Cartesian product phase-spectral measures

Generalized Kolountzakis' Conjecture can be generalized to the phase spectrality of product measures.

Question 4: Let μ and ν be two finite Borel measures on $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, is it true that $\mu \times \nu$ is a phase-spectral measure with orthgonal basis $E(\Lambda, \varphi)$ if and only if both μ and ν are phase-spectral measures?

A function φ or a set Λ has a product structure if $\varphi = \varphi_1 \times \varphi_2$ or $\Lambda = \Lambda_1 \times \Lambda_2$.

We will partially answer Question 4 in the following three cases:
Case 1: Both Λ and φ have product structures;
Case 2: Λ has a product structure, whereas φ doesn't;
Case 3: φ has a product structure, whereas Λ doesn't.

We give a positive answer to Question 4 for **Case 1**.

Theorem (F.-Zhou)

Let μ and ν be two finite Borel measures supported on $K_\mu \subseteq \mathbb{R}^n$ and $K_\nu \subseteq \mathbb{R}^m$ respectively. Let $\varphi_1 : K_\mu \to \mathbb{R}^{d_1}$, $\varphi_2 : K_\nu \to \mathbb{R}^{d_2}$ be Borel measurable and let $\Lambda_1 \subseteq \mathbb{R}^{d_1}$ and $\Lambda_2 \subseteq \mathbb{R}^{d_2}$ be countable sets. Denote $\varphi = \varphi_1 \times \varphi_2$ and $\Lambda = \Lambda_1 \times \Lambda_2$. Then $E(\varphi, \Lambda)$ is an orthogonal basis for $L^2(\mu \times \nu)$ if and only if $E(\varphi_1, \Lambda_1)$ is an orthogonal basis for $L^2(\mu)$ and $E(\varphi_2, \Lambda_2)$ is an orthogonal basis for $L^2(\nu)$.

- $(\mu \times \nu, \Lambda_1 \times \Lambda_2)$ is a spectral pair iff both (μ, Λ_1) and (ν, Λ_2) are spectral pairs.

Cartesian product phase spectral measures

We give a counterexample to Question 4 for **Case 2**.

Counterexample (F.-Zhou)

Let μ be a discrete measure define on $\{0, 1, 2, 3\}$ with non-equal weights $\{\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, \frac{1}{8}\}$, and $d\nu = \mathbf{1}_{[0,1]^2} dx$ Then $\mu \times \nu$ is supported on some pieces of square on $z = \{0, 1, 2, 3\}$. Define

$$\varphi_1 = \begin{cases} (x, y, z) & z = 0 \\ (x, 2y, z) & z = 1 \\ (x, 4y, z) & z = 2 \\ (x, y, z) & z = 3 \end{cases} ; \varphi_2 = \begin{cases} (x, y, z) & z = 0 \\ (x, y, z) + (0, 1, -1) & z = 1 \\ (x, y, z) + (0, 3, -2) & z = 2 \\ (x, y, z) + (0, 7, -3) & z = 3 \end{cases}$$

and $\varphi := \varphi_1 \circ \varphi_2$. Then $\varphi_*(\mu \times \nu)$ is a spectral measure. However, μ is not spectral.

Cartesian product phase spectral measures

Under some additional conditions on μ and ν, we give a positive answer to Question 4 for **Case 3**.

Theorem (F.-Zhou)

For any finite Borel measures μ and ν, if there exists a $\mu-$essentially injective function φ_1 such that $d(\varphi_1 * \mu) = \mathbf{1}_I dx$, where I is a unit cube and a ν-essentially injective function φ_2 such that $\varphi_2 * \nu$ is an absolutely continuous measure. Then $(\varphi_1 \times \varphi_2)_*(\mu \times \nu)$ is a spectral measure iff both $\varphi_{1*}\mu$ and $\varphi_{2*}\nu$ are spectral measures.

- $\mathbf{1}_I dx \times \nu$ with an absolutely continuous ν is a spectral measure iff ν is a spectral measure.

Further remarks

Based on our results, we can furtherly ask "are the following measures phase spectral?"

1.Absolutely continuous measures on $\mathbb{R}^d$, $d \geq 2$ are phase spectral?

2. General Borel probability measure generated by an IFS.

3. For the phase spectrality of $\mu \times \nu$, how about the case that neither Λ nor φ has a product structure?

Thank You